32.14 VECTOR SPACE

(*R.G.P.V., Bhopal, III Semester, Dec. 200*

Let V be a non-empty set and F be the field of real numbers. Let we have two compositions one is plus (+) between two members of V and other is dot $(\cdot)$ between a member of V and member of F. V is said to be vector space if the following properties hold good.

1. Closure property.

$$\forall\ a, b \in R \qquad \Rightarrow \quad a + b \in R.$$

2. Associativity of addition.

For all $\alpha, \beta, \gamma \in V$.

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma.$$

3. Existence of the neutral element.

There exists an element $0 \in V$, such that

$$\alpha + 0 = 0 + \alpha = \alpha, \quad \text{for all } \alpha \in V.$$

4. Existence of additive inverse.

For each $\alpha \in V$, there exists $\beta \in V$, such that

$$\alpha + \beta = \beta + \alpha = 0.$$

5. Commutativity of addition.

For all $\alpha, \beta \in V$, $\quad \alpha + \beta = \beta + \alpha$

6. Closure property.

$\forall\ \alpha, \beta \in R \quad \Rightarrow \alpha \,.\, \beta \in R$

7. Associativity of scalar multiplication.

For all $x, y \in F$ and $\alpha \in V$

$$x\,(y\,\alpha) = (x\,y)\,\alpha.$$

8. Distributivity of scalar multiplication over addition.

For all $x \in F$, $\alpha, \beta \in V$,

$$x\,(\alpha + \beta) = x\,\alpha + x\,\beta.$$

9. Distributivity of scalar multiplication over addition in F.

For all $x, y \in F$, $\alpha \in V$.

$$(x + y)\,\alpha = x\,\alpha + y\,\alpha.$$

10. Property of unity.

If 1 be the identity in F, then for all $\alpha \in V$.

$$1 \cdot \alpha = \alpha.$$

Note. Vectors will be denoted by α, β, γ while scalars will be denoted by a, b, c, d or x, y, z.

Theorem 1. *Let $V(F)$ be a vector space, and*

(i) If α is a non-zero element of V and $a, b \in F$, then

$$a\alpha = b\alpha \;\Rightarrow\; a = b$$

(ii) If a is a non-zero element of F and $\alpha, \beta \in V$, then

$$a\alpha = a\,\beta \;\Rightarrow\; \alpha = \beta.$$

Proof. (*i*) We have, $\quad a\alpha = b\alpha \Rightarrow a\alpha - b\alpha = \bar{0} \quad \Rightarrow (a - b)\,\alpha = \bar{0}$.

$\Rightarrow a - b = 0 \quad [\because\ \bar{\alpha} \neq 0] \quad \Rightarrow a = b.$

Hence, $\quad a\alpha = b\alpha \quad \Rightarrow a = b \qquad [\alpha \neq \bar{0}]$

(*ii*) We have, $\quad a\alpha = a\beta \Rightarrow a\alpha - a\beta = 0 \quad \Rightarrow a\,(\alpha - \beta) = \bar{0}$.

$\Rightarrow \alpha - \beta = \bar{0} \quad [\because\ a \neq 0] \quad \Rightarrow \alpha = \beta.$

Hence, $\quad a\alpha = a\beta \quad \Rightarrow \alpha = \beta \qquad [a \neq 0]$

33.19 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

Vectors (matrices) $X_1, X_2, \ldots\ldots\ldots X_n$ are said to be dependent if

(1) all the vectors (row or column matrices) are of the same order.

(2) n scalars $\lambda_1, \lambda_2 \ldots \lambda_n$ (not all zero) exist such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \ldots\ldots\ldots + \lambda_n X_n = 0$$

Otherwise they are linearly independent.

Example 4. *Show that the vectors* $X_1 = (1, 2, 3)$, $X_2 = (3, -1, 4)$ and $X_3 = (4, 1, 7)$ *are linearly dependent.*

Solution.
$$X_1 = (1, 2, 3)$$
$$X_2 = (3, -1, 4)$$
$$X_3 = (4, 1, 7)$$

If $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$

$$\Rightarrow \quad \lambda_1(1, 2, 3) + \lambda_2(3, -1, 4) + \lambda_3(4, 1, 7) = 0$$

$$\Rightarrow \quad [(\lambda_1 + 3\lambda_2 + 4\lambda_3), (2\lambda_1 - \lambda_2 + \lambda_3), (3\lambda_1 + 4\lambda_2 + 7\lambda_3)] = (0, 0, 0)$$

$$\Rightarrow \quad \lambda_1 + 3\lambda_2 + 4\lambda_3 = 0 \qquad \ldots (1)$$

$$2\lambda_1 - \lambda_2 + \lambda_3 = 0 \qquad \ldots (2)$$

$$3\lambda_1 + 4\lambda_2 + 7\lambda_3 = 0 \qquad \ldots (3)$$

Solving (1), (2) and (3), we get

$$\lambda_1 = 1, \lambda_2 = 1 \text{ and } \lambda_3 = 1$$

So, the vectors are linearly dependent. **Proved.**

Example 5. *Are the vectors*

$X_1 = (1, 0, 0)$, $X_2 = (0, 1, 0)$ *and* $X_3 = (0, 0, 1)$

linearly dependent?

Solution. Consider the matrix equation $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$

$$\Rightarrow \lambda_1(1, 0, 0) + \lambda_2(0, 1, 0) + \lambda_3(0, 0, 1) = 0$$

$$\Rightarrow (\lambda_1 + 0.\lambda_2 + 0.\lambda_3, 0\lambda_1 + \lambda_2 + 0\lambda_3, 0\lambda_1 + 0\lambda_2 + \lambda_3) = (0, 0, 0)$$

$$\Rightarrow (\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$$

As $\lambda_1, \lambda_2, \lambda_3$ all are zero, therefore X_1, X_2, X_3 are linearly independent vectors. **Ans.**

Example 6. *Examine the following vectors for linear dependence and find the r exists*

$$X_1 = (1, 2, 4),\quad X_2 = (2, -1, 3),\quad X_3 = (0, 1, 2),\quad X_4 = (-3, 7, 2)$$

Solution. Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$$

$$\Rightarrow \quad \lambda_1(1,2,4) + \lambda_2(2,-1,3) + \lambda_3(0,1,2) + \lambda_4(-3,7,2) = 0$$

$$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$$

$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$

$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

This is the homogeneous system

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad A\lambda = 0$$

$$R_2 \to R_2 - 2R_1,\ R_3 \to R_2 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$

$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$$

$$\lambda_3 + \lambda_4 = 0$$

Let $\lambda_4 = t$,

$$\lambda_3 + t = 0 \Rightarrow \lambda_3 = -t$$

$$-5\lambda_2 - t + 13t = 0 \Rightarrow \lambda_2 = \frac{12t}{5}$$

$$\lambda_1 + \frac{24t}{5} - 3t = 0 \quad \Rightarrow \quad \lambda_1 = \frac{-9t}{5}$$

Hence, the given vectors are linearly dependent.

Substituting the values of sum in (1), we get

33.21 VECTOR SUB SPACES

Let V be a vector space over a field F, then, a non-empty subset W of V is called a vector subspace of V, if W is a vector space in its own right with respect to the addition and scalar multiplication compositions on V, restricted only on points of W.

Remark. In an arbitrary vector space V, the sets $\{0\}$ and V are clearly subspaces of V and are known as trivial sub-spaces. However, our interest lies in non-trivial subspaces.

Example 20. *Let R be the field of real numbers. Which of the following are subspaces of V_3 (R)?*

(*i*) $W_1 = \{(x, 2y, 3z) : x, y, z \in R\}$ (*ii*) $W_2 = \{(x, x, x) : x \in R\}$

(*iii*) $W_3 = \{(x, y, z) : x, y, z$ *are rational numbers*$\}$

Solution. (*i*) Here $W_1 = \{(x, 2y, 3z) : x, y, z \in R\}$.

Let $\alpha = (x_1, 2y_1, 3z_1)$ and $\beta = (x_2, 2y_2, 3z_2)$ be any two arbitrary elements of W_1, then $x_1, y_1, z_1, x_2, y_2, z_2 \in R$. If $a, b \in R$ be any two real numbers, then

$$a\alpha + b\beta = a(x_1, 2y_1, 3z_1) + b(x_2, 2y_2, 3z_2) = (ax_1 + bx_2, 2ay_1 + 2by_2, 3az_1 + 3bz_2)$$

$$= [ax_1 + bx_2, 2(ay_1 + by_2), 3(az_1 + bz_2)] \in W_1 \qquad [\because ax_1 + bx_2 \text{ etc.} \in R]$$

$\therefore$ $a, b \in R$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$.

Hence, W_1 is a subspace of V_3 (*R*).

(*ii*) Here $W_2 = \{(x, x, x) : x \in R\}$. Let $\alpha = (x_1, x_1, x_1)$ and $\beta = (x_2, x_2, x_2)$ be any two elements of W_2, then $x_1, x_2 \in R$. If $a, b \in R$ be any two real numbers, then we have

$$a\alpha + b\beta = a(x_1, x_1, x_1) + b(x_2, x_2, x_2) = (ax_1 + bx_2, ax_1 + bx_2, ax_1 + bx_2) \in W_2$$

$$[\because ax_1 + bx_2 \in W_2]$$

Hence W_2 is a subspace of V_3 (*R*).

(*iii*) Here $W_3 = \{(x, y, z) : x, y, z$ are rational numbers$\}$.

Let $\alpha = (4, 5, 7)$ be any element of W_3. If $a = \sqrt{6}$ is an element of *R*, then

$a\alpha = \sqrt{6}(4, 5, 7) = (4\sqrt{6}, 5\sqrt{6}, 7\sqrt{6}) \notin W_3$. Since $4\sqrt{6}, 5\sqrt{6}, 7\sqrt{6}$ are not rational numbers.

Consequently, W_3 is not closed with respect to scalar multiplication. Hence, W_3 is not a subspace of V_3 (*R*). **Ans.**

33.28 BASIS (*R.G.P.V., Bhopal, III Semester, Dec. 2006*)

Let V be a vector space. A collection of vectors $\alpha_1, \alpha_2, ...\alpha_r$ is said to form a basis of V if $\alpha_1, \alpha_2, ... \alpha_r$ are linearly independent and if they generate V.

Coordinate of a Vector

Let $V(F)$ be a finite dimensional vector space. Let $B = \alpha_1, \alpha_2, ... \alpha_n$ be ordered basis of V. Let $\alpha \in V$. Then there exists a unique n-tuple $(x_1, x_2, ... x_n)$ of scalars such that

$$\alpha = x_1 \alpha_1 + x_2 \alpha_2 + ... x_n \alpha_n$$

$(x_1, x_2, ... x_n)$ is called coordinates of the basis V.

33.29 DIMENSION OR RANK OF A VECTOR SPACE

(*R.G.P.V., Bhopal, III Semester, Dec. 2005*)

The number of vectors presents in a basis of a vector space V is called the dimension of V. It is denoted by dim (V).

Example 25. *Dimension of the vector space V_4 is 4, since the four vectors (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1) form a basis of V_4.*

Example 26. *Dim $(V_n) = n$, since there are n number of vectors in a basis of V_n.*

Here, we are mainly concern with finite dimensional vector space. The dimension of vector space may be infinite.

Example 27. *Each set of $(n + 1)$ or more vectors of a finite dimensional vector space $V(F)$ of dimension n is :*

(i) linearly dependent   *(ii) a basis of $V(F)$*

(iii) a subspace of $V(F)$ *(iv) linearly independent*

(*R.G.P.V., Bhopal, III Semester, Dec. 2007, 2006*)

Solution. Dimension of vector space $V(F)$ is n, therefore $V(F)$ may have at most n independent vectors. Here the number of vectors are $(n + 1)$, so they are linearly dependent.

Ans.

Example 28. *Show that the vectors* (1, 0, 0), (1, 1, 0), (1, 1, 1) form a basis for R^3.

(*R. G. P. V. Bhopal, III Semester, June 2007*)

Solution. Let $a_1, a_2, a_3 \in R$ be such that

$$a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(1, 1, 1) = 0$$

$$(a_1 + a_2 + a_3) + (0 + a_2 + a_3) + (0 + 0 + a_3) = 0 \qquad ...(1)$$

$$a_1 + a_2 + a_3 = 0$$

$$a_2 + a_3 = 0$$

$$a_3 = 0$$

The matrix of the coefficients of the equations is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank $(A) = 3$

$a_3 = 0,\ a_2 = 0$ and $a_1 = 0$

The non-zero values of a_1, a_2, a_3 do not exist which can satisfy (1).

Thus, (1, 0, 0), (1, 1, 0) and (1, 1, 1) are linearly independent.

Hence, the set of given vectors form a basis of R^3. **Proved.**

Example 29. *Show that $a_1 = (1, 0, 0)$, $a_2 = (0, 1, 0)$, $a_3 = (0, 0, 1)$ form a basis of the vectors space V_3.*

Solution. Let a_1, a_2, a_3 be non-zero real numbers.

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

$$a_1 (1, 0, 0) + a_2 (0, 1, 0) + a_3 (0, 0, 1) = 0 \qquad \dots (1)$$

$$(a_1,\ a_2, a_3) = 0$$

$$\Rightarrow \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0$$

Thus non-zero values of a_1, a_2, a_3 do not exist which can satisfy (1).

Hence, the given system of vectors is linearly independent.

Let $\alpha = x_1 (1, 0, 0) + x_2 (0, 1, 0) + x_3 (0, 0, 1)$

$$= x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3$$

It shows that any vector space V_3 can be expressed as a linear combination of $\alpha_1, \alpha_2, \alpha_3$. So $\alpha_1, \alpha_2, \alpha_3$ are the generators.

Hence, $\alpha_1, \alpha_2, \alpha_3$ form basis of V. **Proved.**

Example 30. *Determine whether the following vectors form a basis of R^3 or not*

(1, 1, 2), (1, 2, 5), (5, 3, 4).

Solution. We know that dim $R^3 = 3$. Thus if the given set of vectors is linearly independent, then it will be a basis of R^3 otherwise not.

Now, for $a_1, a_2, a_3 \in R$

$$a_1 (1, 1, 2) + a_2 (1, 2, 5) + a_3 (5, 3, 4) = (0, 0, 0)$$

$$\Rightarrow \quad (a_1 + a_2 + 5a_3, a_1 + 2a_2 + 3a_3, 2a_1 + 5a_2 + 4a_3 = (0, 0, 0)$$

$$\therefore \quad a_1 + a_2 + 5a_3 = 0 \qquad \dots (1)$$

$$a_1 + 2a_2 + 3a_3 = 0 \qquad \dots (2)$$

$$2a_1 + 5a_2 + 4a_3 = 0 \qquad \dots (3)$$

$\therefore$ The coefficient matrix of these equations is

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$$

Here $|A| = 1 (8 - 15) - 1 (4 - 6) + 5 (5 - 4) = -7 + 2 + 5 = 0$

$$\text{Rank of } A \neq 3$$

$$a_1 = a_2 = a_3$$

The scalars a_1, a_2, a_3 are not all zero, therefore, the given set S of vectors is linearly dependent and hence the given set of vectors are not basis set. **Ans.**

Example 32. *Show that the set $S = \{(1, 0, 0), (1, 1, 0)\ (1, 1, 1), (0, 1, 0)\}$ is not a basis set.*

Solution. Let $a_1, a_2, a_3, a_4 \in R$ be such that

$$a_1\,(1, 0, 0) + a_2\,(1, 1, 0) + a_3\,(1, 1, 1) + a_4\,(0, 1, 0) = 0$$

$$\Rightarrow \quad (a_1 + a_2 + a_3 + 0a_4,\ 0a_1 + a_2 + a_3 + a_4, 0a_1 + 0a_2 + a_3 + 0a_4) = (0, 0, 0)$$

$$\Rightarrow \quad a_1 + a_2 + a_3 = 0,\ a_2 + a_3 + a_4 = 0, \quad a_3 = 0$$

The coefficient matrix of these equations are in the matrix form.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here, $\quad |A| = 1\,(1 - 0) = 1 \neq 0$

$$\Rightarrow \quad R\,(A) = 3$$

$\therefore$ The scalars a_1, a_2, a_3, a_4 are not all zero, therefore, the given set S of vectors is linearly dependent and hence S is not a basis set. **Proved.**

34.1 LINEAR TRANSFORMATIONS

Let U and V be two vector spaces over the same field F. Then, a mapping T of U into V is called a linear transformation or a homomorphism of U into V, if

(*i*) $T(\alpha + \beta) = T(\alpha) + T(\beta)\ \forall \alpha, \beta \in U$ and (*ii*) $T(a\alpha) = a\,T(\alpha)\ \forall a \in F, \alpha \in U.$

The conditions (*i*) and (*ii*) above can be combined as

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)\ \forall a, b \in F,\ \forall \alpha, \beta \in U$$

A transformation of U into itself is called a linear operator.

A one-one linear transformation of U onto V is called an isomorphism. In case, there exist an isomorphism of U onto V, we say that U is isomorphic to V and we write, $U \cong V$.

(*i*) Zero Transformation. If $U(F)$ and $V(F)$ be two vector spaces over the same field F, then the mapping $\hat{0}: U \to V$ defined by $\hat{0}(x) = 0, \forall x \in U$ is said to be **zero transformation.** $\bar{0}$ **is called zero operator.**

(ii) Identity Transformation (or Identity operator). If $V(F)$ is a vector space, then the mapping $T: V \to V$ defined by $T(v) = v\ \forall\ v \in V$ is called an **identity transformation.** T is called identity operator.

Theorem 1. *To prove that zero operator is linear operator.*

Proof. Let $U(F)$ and $V(F)$ be two vector spaces over the same field F. Let $\hat{0}: U \to V$ defined by $\hat{0}(x) = \hat{0}\ \forall\ x \in U$ be the zero operator.

Let $\alpha, \beta \in U$ and $a, b \in F$, then $\hat{0}(a\alpha + a\beta) = 0 = 0 + 0 = a\hat{0}(\alpha) + b\hat{0}(\beta)$.

Hence, $\hat{0}$ is a linear operator. **Proved.**

Theorem 2. *To prove that identity operator is linear operator.*

Proof. Let T be the identity operator on $V(F)$. Then $T(x) = x, \forall\ x \in V.$

Let $\alpha, \beta \in V$ and $a, b \in F$, then $a\alpha + b\beta \in V$

Now, $T(a\alpha + b\beta) = a\alpha + b\beta$ {by definition of T}

$= aT(\alpha) + bT(\beta)$ [$\because \alpha = T(\alpha), \beta = T(\beta)$]

Hence, T is a linear operator. **Proved.**

Example 1. *Show that the translation mapping $f: V_2(R) \to V_2(R)$ defined by $f(x, y) = (x + 2, y + 3)$ is not linear.*

Solution. Here $\bar{0} = (0, 0)$ is the zero vector of $V_2(R)$. Thus, by definition of f, we have
$f(0) = f(0, 0) = (0 + 2, 0 + 3) = (2, 3) \neq 0$.
Since f does not map the zero vector onto the zero vector, hence f is not linear.

Proved

Example 2. *Prove that: $T : R^2 \to R^2$*
$T(x_1, x_2) = (x_1, 0)$, is a linear transformation.

Solution. Let $a, b \in R$ and $x \equiv (x_1, x_2)$ and $y \equiv (y_1, y_2) \in R^2$

We see $T = (a x + b y) = T\{a(x_1, x_2) + b(y_1, y_2)\}$

$= T\{(a x_1, a x_2) + (b y_1, b y_2)\}$

$= T\{(a x_1 + b y_1; a x_2 + b y_2)\}$

$= (ax_1 + by_1, 0)$

$= (ax_1, 0) + (by_1, 0)$

$= a(x_1, 0) + b(y_1, 0)$

$= aT(x_1) + bT(y_1)$.

Proved

So, T is a linear transformation.

Example 3. *Show that the mapping $f : V_2(R) \to V_3(R)$ defined by*
$f(a, b) = (a, b, 0)$ is a linear transformation.
(R.G.P.V. Bhopal, III Semester, Dec. 2007)

Solution. Let $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2) \in V_2(R)$

If $a, b \in F$, then $f(a\alpha + b\beta) = f[a(a_1, b_1) + b(a_2, b_2)]$

$= f[(aa_1, ab_1) + (ba_2, bb_2)]$

$= f(aa_1 + ba_2, ab_1 + bb_2)$

$= (aa_1 + ba_2, ab_1 + bb_2, 0)$

$= a(a_1, b_1, 0) + b(a_2, b_2, 0)$

$= a f(\alpha) + b f(\beta)$

Proved

Hence, f is a linear transformation.

Example 4. *Show that the mapping $f: V_3(R) \to V_2(R)$ defined by $f(a, b, c) = (c, a + b)$ is a linear transformation.*

Solution. Let $\alpha = (a_1, b_1, c_1)$, $\beta = (a_2, b_2, c_2) \in V_3(R)$. If $a, b, c \in R$, then

$f(a\alpha + b\beta) = f[a(a_1, b_1, c_1) + b(a_2, b_2, c_2)] = f[(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)]$

$= (ac_1 + bc_2, aa_1 + ba_2 + ab_1 + bb_2)$ [By definition of f]

$= (ac_1, aa_1 + ab_1) + (bc_2, ba_2 + bb_2) = a(c_1, a_1 + b_1) + b(c_2, a_2 + b_2)$

$= af(a_1, b_1, c_1) + bf(a_2, b_2, c_2) = af(\alpha) + bf(\beta)$

Proved

Hence, f is a linear transformation.

Example 5. *Show that the mapping $f : V_2(R) \to V_2(R)$ defined by $f(x, y) = (x^3, y^3)$ is not a linear transformation.*

Solution. Let $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2) \in V_2(R)$, then

$f(\alpha + \beta) = f[(x_1, y_1) + (x_2, y_2)] = f[(x_1 + x_2, y_1 + y_2)] = [(x_1 + x_2)^3, (y_1 + y_2)^3]$

$\neq \left(x_1^3 + x_2^3, y_1^3 + y_2^3\right)$

$\neq \left(x_1^3, y_1^3\right) + \left(x_2^3, y_2^3\right)$

$\neq f(\alpha) + f(\beta)$

Hence, f is not a linear transformation.

Proved

Example 6. *Show that the mapping $f: V_3(R) \to V_2(R)$ defined by $f(a, b, c) = (a - b, a + c)$ is linear transfromation.* (*R.G.P.V., Bhopal, III Semester, Dec. 2006*)

Solution. Let $\alpha, \beta \in V_3$

$$\alpha = (a_1, b_1, c_1),\ \ \beta = (a_2, b_2, c_2)$$

$$\begin{aligned} f(\alpha + \beta) &= f[(a_1, b_1, c_1) + (a_2, b_2, c_2)] \\ &= f[(a_1 + a_2, b_1 + b_2, c_1 + c_2] \\ &= f[a_1 + a_2 - b_1 + b_2,\ \ a_1 + a_2, c_2 + c_2)] \\ &= f[a_1 - b_1, + a_2 + b_2 + a_1 + c_1, + a_2 + c_2)] \\ &= (a_1 - b_1, a_1 + c_1) + (a_2 + b_2, a_2 + c_2) \\ &= f[(a_1, b_1, c_1) + f(a_2, b_2, c_2) \\ &= f(\alpha) + f(\beta) \end{aligned}$$

For any real number

$$\begin{aligned} f(k \cdot \alpha) &= f[k(a_1, b_1, c_1)] \\ &= f(ka_1, kb_1, kc_1) \\ &= (ka_1 - kb_1, ka_1 + kc_1) \\ &= k(a_1 - b_1, a_1 + c_1) \\ &= kf(a_1, b_1, c_1) \\ &= kf(\alpha) \end{aligned}$$

So, f is a linear transformation from V_3 to V_2. **Proved.**

Example 7. Show that the function T ...

34.2 MATRIX OF A LINEAR TRANSFORMATION

Consider the simultaneous equations given below:

$$2x_1 + 3x_2 - x_3 = 2$$
$$5x_1 - 6x_2 - 3x_3 - 10$$
$$x_1 + x_2 + x_3 = 8$$

The left hand side of the equations can be considered as the linear transformations of T

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 & + & 3x_2 & - & x_3 \\ 5x_1 & + & 6x_2 & - & 3x_3 \\ x_1 & + & x_2 & + & x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

we can write the formula $A{:}T_A(X) = AX$

In general for $m \times n$ matrix the transformation is $TA : R^n \rightarrow R^m$

such transformation is called matrix transformation.

For example. the matrix $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 7 & 9 \\ 4 & 1 & 2 \end{bmatrix}$ gives matrix transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & + & 5x_2 & + & 3x_3 \\ 2x_1 & + & 7x_2 & + & 9x_3 \\ 4x_1 & + & x_2 & + & 2x_3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 7 & 9 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This can also be written horizontally

$$T : R^3 \to R^3, [x_1, x_2, x_3] \to [x_1 + 5x_2 + 3x_3,\ \ 2x_1 + 7x_2 + 9x_3,\ \ 4x_1 + x_2 + 2x_3]$$

This transformation is not matrix transformation because it can not be expressed as $A\ X$ for constant matrix A.

Example 11. *The matrix of linear mapping $T : R^3 \to R^3$ given by $T(a, b, c) = (a, b, 0)$ relative to standard basis is*

(*i*) $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(*R.G.P.V., Bhopal, III Semester, Dec. 2006*)

Solution. The standard basis of R^3 is $B = (e_1, e_2, e_3)$, where

$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$

Thus by definition of T, we have

$$T(e_1) = T(1, 0, 0) = (1, 0, 0) = 1\,(1, 0, 0) + 0\,(0, 1, 0) + 0\,(0, 0, 1)$$
$$= 1\,e_1 + 0e_2 + 0e_3$$
$$T(e_2) = T(0, 1, 0) = (0, 1, 0) = 0\,(1, 0, 0) + 1\,(0, 1, 0) + 0\,(0, 0, 1)$$
$$= 0e_1 + 1.e_2 + 0.e_3$$
$$T(e_3) = T(0, 0, 1) = (0, 0, 1) = 0\,(1, 0, 0) + 0\,(0, 1, 0) + 1\,(0, 0, 1)$$
$$= 0e_1 + 0e_2 + e_3$$

Hence, the coefficient matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and its transpose

matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$[T, B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans. (iii) is correct

Function or Mapping

Let there be two non-empty sets X and Y and there is some rule or correspondence which assigns to each element $x \in X$, a unique element $y \in Y$, then this rule or correspondence is said to be a *mapping* or a *function* and denoted by f, *i.e.*, $f : X \to Y$ and read as 'f is a function of X to Y' or f is a 'mapping of X to Y'.

The set X is called the *domain* of the given function f and the set Y of all the values assumed by it is called its *Range* or *Image set*. Also Y is called the *co-domain* of f.

y is sometimes known as image of x and written as $y = f(x)$. Here $f(x)$ is read as '*image of x under the rule f*' or simply 'f of x'. The rule f is also known as *mapping* or *transformation* or *operator* and x is also known as *pre-image* of y.

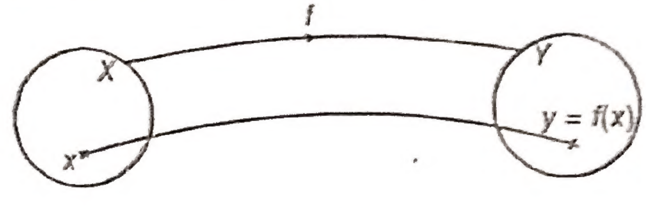

Fig. 4.3

A function whose range has a single element is said to be *constant function.*

Diagrammatical representation of $y = f(x)$ with a rule f defined by $x \to f(x)$ is shown in Fig. 4.3.

If $y = x^2$, then the rule f is $x \to x^2$

which is shown in Fig. 4.4 for positive integral values of x.

Fig. 4.4

Functions defined as sets of ordered pairs. Given two non-empty sets X and Y, a function f from X to Y is a subset of $X \times Y$ provided

(*i*) $\forall\ x \in X$, $(x, y) \in f$ for some $y \in Y$, *i.e.*, $\exists$ (*there exists*) a rule f so that every element of X has image.

(*ii*) $(x, y) \in f$ and $(x, y') \in f \Rightarrow y = y'$, *i.e.*, the image is unique.

The *graph* of f is defined as the subset of $X \times Y$ given by $\{[x, f(x)] : x \in X\}$ and that *range* of f as the set of all images under f given by $f[X] = \{y \in Y : y = f(x)$ for some $x \in X\} = \{f(x) : x \in X\}$.

In case $A \subset X$ then the set $\{f(x) : x \in A\}$ is known as the *image* of A under f and denoted by $f[A]$. Also if $B \subset Y$, then the set $\{x \in X : f(x) \in B\}$ is known as the *inverse image* of B under f and denoted by $f^{-1}[B]$.

Extension and Restriction of a function. Given two functions f and g such that f contains the domain of g and $f(x) = g(x)\ \forall\ x$ in the domain of g, the function f is said to be the *extension* of g and g is said to be the *restriction* of f.

Real and Complex functions. If range of f consists of real numbers, f is said to be a *real function* and if its range consists of complex numbers, f is said to be a *complex function*.

Onto and Into Mappings. If the range is completely filled up, the mapping is said to be *onto* and if the range is not completely filled up then it is *into*. In other words, if $\exists$ at least one $y \in Y$ which is not an $f(x)$ for any $x \in X$, then the mapping f is said to be *into* otherwise it is said to be *onto* or *surjective*. The surjective function is also known as a *surjection* or an *epimorphism*.

One-one and Many-one Mappings. Given two non-empty sets X and Y, if two different elements in X always have different images under the rule f, then f is said to be a *one-one mapping* or an *injection* or *monomorphism* of X into (onto) Y and if the two or more different elements of X have the same image under f, then f is said to be a *many-one mapping* of X into (onto) Y.

Diagrammatical representation of such functions are shown in Figs. 4.5, 4.6, 4.7, 4.8.

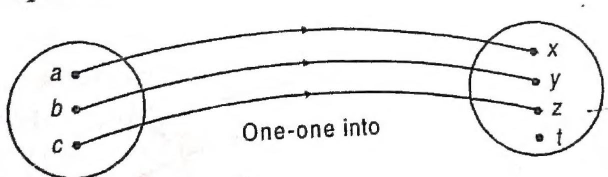

Fig. 4.5

A function which is both surjective and injective is known as *bijective*, *i.e.*, a one-one onto mapping is also known as a *bijection* and a bijection of a set X onto itself is known as **Permutation** of X.

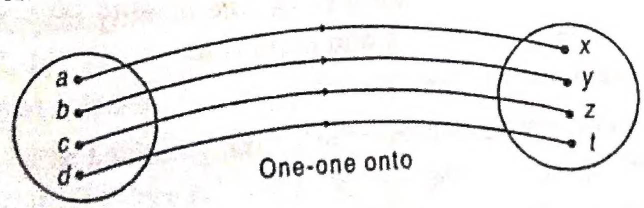

Fig. 4.6

If $f : X \to Y$, f is one-one if $x_1 \neq x_2 \Rightarrow f_1(x_1) \neq f(x_2)\ \forall\ x_1, x_2 \in X$. In case f is *into*, the range of f is proper subset of Y, *i.e.*, $f[X] \subset Y$ and $f[X] \neq Y$.

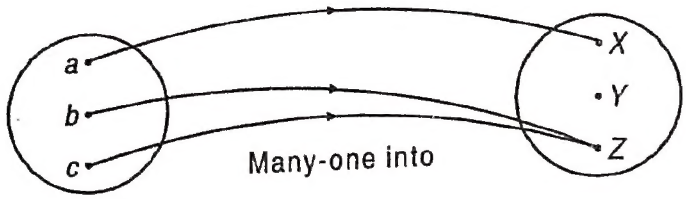

Fig. 4.7

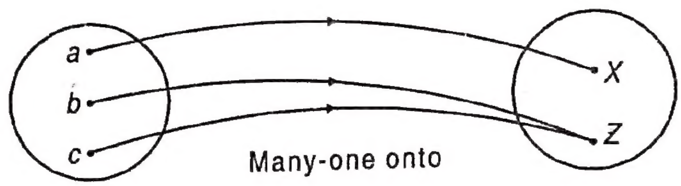

Fig. 4.8

In case f is *onto*, the range of f is equal to Y, *i.e.*, $f[X] = Y$.

Inverse mapping. Let f represent a function (mapping) which is both *onto* and *one-one* defined as $f: X \rightarrow Y$, then its inverse mapping $f^{-1}: Y \rightarrow X$ is defined as below:

$\forall y \in Y$, if we find the unique element $x \in X$ s.t. $f(x) = y$ then x is defined to be $f^{-1}(y)$, *i.e.*, $f^{-1}(y) = \{x : x \in X, f(x) = y\}$ which follows that $f^{-1}(y)$ is always a subset of X.

Diagrammatical representation of an inverse mapping is shown in Fig. 4.9.

One-one onto mapping is often called as *one-to-one correspondence.* Thus if f is a one-to-one correspondence between X and Y, then f^{-1} is a one-to-one correspondence between Y and X.

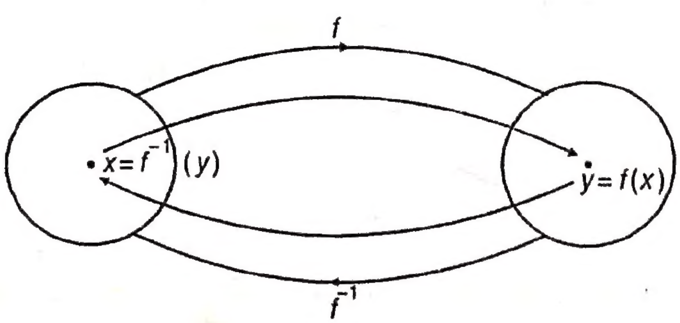

Fig. 4.9